Finitely Axiomatized Set Theory: a nonstandard first-order theory implying ZF

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Abstract — It is well-known that a finite axiomatization of Zermelo-Fraenkel set theory (ZF) is not possible in the same first-order language. In this note we show that a finite axiomatization is possible if we extent the language of ZF with the new logical concept of 'universal quantification over a family of variables indexed in an arbitrary set X'. We axiomatically introduce Finitely Axiomatized Set Theory (FAST), which consists of eleven theorems of ZF plus a new constructive axiom called the family set axiom (FAM); the latter is a generalization of the pair axiom of ZF, and uses the new concept of quantification. We prove that FAM enables to derive the axioms schemes of separation and substitution of ZF from FAST. The conclusions are (i) that FAST is a finite, nonstandard first-order theory, and (ii) that FAST implies ZF.

1 Introduction

The most widely accepted foundational theory for mathematics is Zermelo-Fraenkel set theory (ZF). While a detailed exposition can be found in the literature, e.g. [1], let's recall that ZF contains seven axioms plus two infinite axiom schemes: the seven axioms are the Extensionality Axiom (EXT), the Empty Set Axiom (EMPTY), the Pair Axiom (PAIR), the Sumset Axiom (SUM), the Powerset Axiom (POW), the Infinite Set Axiom (INF), and the Axiom of Regularity (REG); the two axiom schemes are the Separation Axiom Scheme (SEP) and the Substitution Axiom Scheme (SUB). Altogether, ZF is thus an infinitely axiomatized theory; in addition to that, a well-known theorem of Montague states that a finite axiomatization of ZF is not possible in the language of ZF without assuming extra objects, cf. [2].

As an alternative to ZF, several theories have been suggested that can be finitely axiomatized. No attempt will be made to give a complete overview *hic et nunc*, but examples of such finitely axiomatized alternatives are Von Neumann-Gödel-Bernays set theory (NGB), cf. [3], and Cantor-Von Neumann set theory (CVN), cf. [4]. These two theories, however, either depart from the ontology of ZF or use nonconstructive axioms[1]: NGB uses classes and CVN starts with an axiom stating that there is a set including all sets. Alternative theories will not be discussed: the purpose of this note is purely to show that, while retaining the adage "everything is a set" of ZF, a finite axiomatization of set theory *is* possible with a new **constructive** axiom which uses a new concept of universal quantification.

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[1] A **constructive axiom** is an axiom that, when certain things are given (e.g. one or two sets or a set and a predicate), states the existence of a uniquely determined other set [5].

The key idea is rather simple and can be explained as follows. In the language of ZF, we can easily formalize the idea of quantification over an n-tuple of variables ranging over sets. This yields well-known expressions like

$$\forall u_1, u_2, \ldots, u_n \Phi \tag{1}$$

which can be read as: for any n-tuple of sets $u_1, u_2, \ldots, u_n$, Φ. Such expressions are defined in the language of ZF by the postulate of meaning

$$\forall u_1, u_2, \ldots, u_n \Phi \Leftrightarrow \forall u_1 \forall u_2 \ldots \forall u_n \Phi \tag{2}$$

Conceptually we can view quantification over an n-tuple of variables $u_1, u_2, \ldots, u_n$ as quantification over a family of variables u_i indexed in the finite set $\{1, 2, \ldots, n\}$, denoted by $(u_i)_{i \in \{1,2,\ldots,n\}}$. The point is, however, that in the language of ZF it is not possible to quantify over an *infinite* family, because the right-hand side of (2) has to be a finite formula. The key idea is then to remove this restriction, and to generalize the idea formalized in (1) into quantification over a family of variables indexed in an arbitrary set X; for a given set X, this yields an expression like

$$\forall (u_i)_{i \in X} \Phi \tag{3}$$

This has to be read as: for any family of sets u_i indexed in X, Φ. However, no postulate of meaning such as expression (2) can be given, since the set X may be infinite: the quantification over a family of sets u_i indexed in a set X is thus a primitive concept, which has no equivalent in the language of ZF.

It is emphasized that conceptually, quantification over a family of variables $\forall (u_i)_{i \in X} \ldots$ does not entail a departure of the ontology of ZF: we can thus consider quantification (3) **without** assuming new objects. In ZF the situation is similar; e.g. in PAIR—when formalized as an expression of the type $\forall x_1, x_2 \Psi$ using definition (2) above—one considers quantification over a two-tuple of variables x_1 and x_2 without viewing the two-tuple as a new object in itself.

The next section introduces Finitely Axiomatized Set Theory (FAST) axiomatically; the final section demonstrates that FAST implies ZF by deriving the schemes SEP and SUB from FAST, and presents the conclusions.

2 Axiomatic introduction of FAST

The universe of discourse is the universe of sets—all terms are sets. Regarding the formal language, the following three clauses have to be *added* to the definition of the syntax of the language of ZF:

- for every constant $\mathbf{x}$, we have variables $a_{\mathbf{x}}, b_{\mathbf{x}}, \ldots$ ranging over sets; here the subscript $\mathbf{x}$ is the **label**;
- we have generic variables $a_i, b_j, \ldots$ of which the subscripts are the label; by an interpretation of the label i as a constant $\mathbf{x}$, a generic variable u_i becomes a variable $u_{\mathbf{x}}$ ranging over sets as above.
- if Φ is a formula, u_i is a generic variable, and X is a term, then $\forall (u_i)_{i \in X} \Phi$ is a formula.

As said, the formula $\forall (u_i)_{i \in X} \Phi$ has to be read as: for any family of sets u_i indexed in X, Φ. A quantification $\forall (u_i)_{i \in X} \ldots$ has thus to be seen as a *simultaneous* quantification over *all* those variables $u_{\mathbf{x}}$ of which the label is a constant of the set X.

Proceeding with the axioms of FAST, the first ten axioms are the following theorems of ZF (formalization omitted):

(i) **Axiom of Extensionality** (EXT): Two sets X and Y are identical if they have the same elements.

(ii) **Empty Set Axiom** (EMPTY): There exists a set $X = \emptyset$ who has no elements.

(iii) **Sum Set Axiom** (SUM): For every set X there exists a set $Y = \bigcup X$ made up of the elements of the elements of X.

(iv) **Powerset Axiom** (POW): For every set X there is a set $Y = \mathcal{P}(X)$ made up of the subsets of X.

(v) **Infinite Set Axiom** (INF): There exists a set that has the empty set as element, as well as the successor $\{x\}$ of each element x.

(vi) **Axiom of Regularity** (REG): Every nonempty set X contains an element Y that has no elements in common with X.

(vii) **Difference Set Axiom** (DIFF): For every pair of sets X and Y there is a set $Z = X - Y$ such that the elements of Z are precisely those elements of X that do not occur in Y.

(viii) **Product Set Axiom** (PROD): For any two nonempty sets X and Y there is a set $Z = X \times Y$ made up of all ordered two-tuples[2] $\langle x, y\rangle$ of which x is an element of X and y an element of Y.

(ix) **Image Set Axiom** (IM): For any function f on[3] a set X, there is a set $Z = f[X]$ made up of precisely those elements y, for which there is an element $x \in X$ such that $\langle x, y\rangle \in f$.

(x) **Reverse Image Set Axiom** (REV): For any function f on a set X and for any element y, there is a set $Z = f^{-1}(y)$ made up of precisely those $x \in X$ such that $\langle x, y\rangle \in f$.

We have then arrived at the key axiom of FAST; in the next section we show that this generalization of ZF's PAIR is so powerful, that the infinite axiom schemes SEP and SUB of ZF can be derived from FAST.

Axiom 2.1. Family Set Axiom (FAM):
$\forall X \forall (u_i)_{i\in X} \exists Z \forall y (y \in Z \Leftrightarrow \exists i \in X (y = u_i))$

Axiom 2.1 guarantees that for any nonempty set X and for any family of sets u_i indexed in X there is a set Z made up of precisely the family of sets u_i. On account of EXT the set Z is unique, and can be denoted by $Z = \{u_i | i \in X\}$.

The point is this: suppose we have, for a given constant (set) $\mathbf{X}$, constructed a uniquely determined set $\mathbf{u_x}$ for every constant $\mathbf{x} \in \mathbf{X}$. That is, suppose we have constructed a family of sets $(\mathbf{u}_i)_{i\in\mathbf{X}}$. Then we have **not yet** constructed the set that contains this family of sets indexed in $\mathbf{X}$, $(\mathbf{u}_i)_{i\in\mathbf{X}}$, as elements. But FAM then guarantees that this set exists—*regardless how this family of sets* $(\mathbf{u}_i)_{i\in\mathbf{X}}$ *is constructed!*

[2] A two-tuple $\langle x, y\rangle$ is a set: $\langle x, y\rangle := \{x, \{x, y\}\}$.

[3] A function f on a set X is a set f made up of precisely one two-tuple $\langle x, y\rangle$ for every element $x \in X$. Every element g of a function space Y^X is thus a function on X.

With the above, all non-logical axioms of FAST have been introduced. However, since quantification over a family of variables indexed in a set X is a new concept in logic, at least the logical axioms for the elimination of the quantifiers from FAM must be given to derive theorems. The two required axioms are straightforward:

Axiom 2.2. First Elimination Axiom (EL1):
$\forall X \forall (u_i)_{i\in X} \exists Z \forall y (y \in Z \Leftrightarrow \exists i \in X (y = u_i)) \Rightarrow$
$\forall (u_i)_{i\in \mathbf{X}} \exists Z \forall y (y \in Z \Leftrightarrow \exists i \in \mathbf{X} (y = u_i))$ for any constant $\mathbf{X}$

Axiom 2.3. Second Elimination Axiom (EL2):
$\forall (u_i)_{i\in \mathbf{X}} \exists Z \forall y (y \in Z \Leftrightarrow \exists i \in \mathbf{X} (y = u_i)) \Rightarrow \exists Z \forall y (y \in Z \Leftrightarrow \exists i \in \mathbf{X} (y = \mathbf{u}_i))$
for any family of constants $(\mathbf{u}_i)_{i\in \mathbf{X}}$

The following example then demonstrates how PAIR of ZF follows from of FAM.

Example 2.4. By universal quantifier elimination EL1 the following expression, which is a theorem of FAST, follows from axiom 2.1 for the case $\mathbf{X} = \{1, 2\}$:

$$\forall (u_i)_{i\in\{1,2\}} \exists Z \forall y (y \in Z \Leftrightarrow \exists i \in \{1, 2\} (y = u_i)) \tag{4}$$

So suppose that we have constructed two sets, $\mathbf{X}_1$ and $\mathbf{X}_2$. In other words: suppose we have constructed a family of sets $(\mathbf{X}_i)_{i\in\{1,2\}}$. Then we do **not** have them in a set yet. None of the first eleven axioms of FAST can put these $\mathbf{X}_i$'s in a set; however, on account of logical axiom EL2 we now derive from theorem (4):

$$\exists Z \forall y (y \in Z \Leftrightarrow \exists i \in \{1, 2\} (y = \mathbf{X}_i)) \tag{5}$$

Of course, formula (5) is equivalent to $\exists Z \forall y (y \in Z \Leftrightarrow y = \mathbf{X}_1 \vee y = \mathbf{X}_2)$, so theorem (4) is then equivalent to PAIR of ZF: $\forall u_1 \forall u_2 \exists Z \forall y (y \in Z \Leftrightarrow y = u_1 \vee y = u_2)$.

This concludes the axiomatic introduction of FAST. We proceed with its discussion in the next section.

3 Discussion and conclusions

In this section we will first derive the infinite schemes SEP and SUB of ZF from FAST. After that, we address the main concern about consistency of the theory. In the remainder of the text, we will, for functions f, use the notation $f : x \mapsto y$ for $\langle x, y\rangle \in f$.

Theorem 3.1. Main Theorem of FAST:
If there is a functional relation $\Phi(x, y)$ that relates every x of a nonempty set X to precisely one y, then there is a function f on X that maps every $x \in X$ to precisely that y for which $\Phi(x, y)$. In a formula:

$$\forall X (\forall x \in X \exists! y \Phi(x, y) \Rightarrow \exists f \forall x \in X \exists! y (f : x \mapsto y \Leftrightarrow \Phi(x, y)))$$

Proof. Suppose for every $x \in X$ we have precisely one y such that $\Phi(x, y)$. Thus, on account of PROD, there is a set $u_x = \{\langle x, y\rangle\} = \{x\} \times \{y\}$ for every $x \in X$; here the singletons $\{x\}$ and $\{y\}$ exist on account of FAM, and y is the element that is in the functional relation $\Phi(x, y)$. On account of FAM, there is then a set $Z = \{u_x | x \in X\}$. On account of SUM, there is then a set $f = \bigcup\{u_x | x \in X\}$. This set f is the requested function f on X. □

Theorem 3.1 is an infinite scheme, with one formula for every functional relation Φ. The point is this: if we have constructed a functional relation $\Phi(x, y)$ that holds for every $x \in X$, then we have **not yet** constructed the (functional) set f containing the two-tuples of the related x's and y's. But this theorem guarantees that this set f exists. So, the practical meaning is this: in the framework of FAST, giving a function prescription **is** constructing a set.

We are now ready to derive the infinite axiom scheme SEP of ZF from the axioms of FAST. For that matter, we have to prove that the following theorem holds for any formula Φ:

Theorem 3.2. Separation Axiom Scheme of ZF:
$\forall X \exists Y \forall z(z \in Y \Leftrightarrow z \in X \wedge \Phi(z))$

Proof. Let the function τ on X be defined by $\begin{cases} \tau : x \mapsto 1 & if\ \Phi(x) \\ \tau : x \mapsto 0 & if\ \neg\Phi(x) \end{cases}$. On account of Theorem 3.1, this function τ exists. On account of REV, the reverse image set $\tau^{-1}(1)$ then exists. This is precisely the set Y requested. □

We proceed by deriving the infinite axiom scheme SUB of ZF from the axioms of FAST. For that matter, we have to prove that the following theorem holds for any set X and for any functional relation Φ:

Theorem 3.3. Substitution Axiom Scheme of ZF:
$\forall x \in X \exists! y \Phi(x, y) \Rightarrow \exists Z \forall u(u \in Z \Leftrightarrow \exists v(v \in X \wedge \Phi(v, u)))$

Proof. Suppose for every $x \in X$ we have precisely one y such that $\Phi(x, y)$. Then on account of Theorem 3.1, there is a function τ on X given by the function prescription $\tau : x \mapsto y \Leftrightarrow \Phi(x, y)$. Thus, on account of IM, there is an image set $\tau[X]$: this is precisely the set Z of theorem 3.3. □

It is herewith proven that the axiom schemes SEP and SUB from ZF follow from FAST. We proceed by addressing the main concern regarding inconsistency, which is that the existence of a set of all sets can be derived from FAST. For that matter, we first prove that the Löwenheim-Skolem theorem does not hold for FAST.

Proposition 3.4.
If FAST has a model $\mathcal{M}$, then $\mathcal{M}$ is uncountable.

Proof. Suppose FAST has a model $\mathcal{M}$, and $\mathcal{M}$ is countable. That means that there are only countably many subsets of $\mathbf{N} = \{0, 1, 2, \ldots\}$ in $\mathcal{M}$, and that the powerset $\mathcal{P}(\mathbf{N})$ in $\mathcal{M}$ contains those subsets: we thus assume that there are subsets of $\mathbf{N}$ that are "missing" in $\mathcal{M}$. Let $\mathbf{A}$ be any subset of $\mathbf{N}$ that is **not** in $\mathcal{M}$, and let $\mathbf{h} \in \mathbf{A}$. Since all numbers $0, 1, 2, \ldots$ are in $\mathcal{M}$, we have the variables $u_0, u_1, u_2, \ldots$ according to the syntax of our language (see Sect. 2). Since $\mathbf{N}$ is in $\mathcal{M}$, we get on account of FAM and the logical axiom EL1 that the formula $\forall (u_i)_{i \in \mathbf{N}} \exists Z \forall y(y \in Z \Leftrightarrow \exists i \in \mathbf{N}(y = u_i))$ holds in $\mathcal{M}$. It should be realized that this is a universal quantification over countably many variables, all ranging over all constants in $\mathcal{M}$. An instantiation is thus that, for a constant $\mathbf{n} \in \mathbf{N}$, to a variable $u_{\mathbf{n}}$ the constant value $\mathbf{u_n}$ is assigned, such that $\mathbf{u_n} = \mathbf{n}$ if $\mathbf{n} \in \mathbf{A}$ and $\mathbf{u_n} = \mathbf{h}$ if $\mathbf{n} \notin \mathbf{A}$. Then on account of the logical axiom EL2 there is a set $\mathbf{Z} = \{\mathbf{u}_n | n \in \mathbf{N}\}$ in $\mathcal{M}$ that is made up of precisely that family of sets $(\mathbf{u}_i)_{i \in \mathbf{N}}$. But $\mathbf{Z}$ is precisely the set $\mathbf{A}$, so $\mathbf{A}$ is in $\mathcal{M}$, contrary to what was assumed. Ergo, if FAST has a model, it is uncountable. □

The second worry is then that the universe of FAST itself contains a set of all sets. That, however, is not the case because FAST contains the Axiom of Regularity. The point is that one must first have a set X before one can construct a family of sets indexed in X: the set X has to be a regular set, and therefore FAM doesn't allow the construction of a set of all sets. We can prove that formally as a theorem:

Theorem 3.5.
$\forall X \forall (u_i)_{i\in X} \neg\exists Z(Z = \{u_i | i \in X\} \wedge \forall y(y \in Z))$

Proof. Suppose there is a regular set X, such that the entire universe of sets can be seen as a family of sets u_i indexed in X. That is, suppose there is a regular set X and a family of sets $(u_i)_{i\in X}$, such that the set Z of that family of sets is the universe of sets. Since we then have $\forall y(y \in Z)$, we then have in particular $Z \in Z$. But that is not possible on account of REG. Thus, there is no such regular set X. □

In the foregoing we have shown that every nonlogical axiom of ZF is a theorem of FAST. That means that every set, which can be constructed with ZF, can also be constructed with FAST: the first conclusion is then that FAST implies ZF. There is then the obvious risk that FAST is not (relatively) consistent, but in any case we have proven that FAST doesn't allow the construction of a set of all sets.

The concept of universal quantification over a family of variables indexed in an arbitrary set X in axiom 2.1 means a departure from the language of ZF, but on the other hand there is no higher-order quantification: the second conclusion is then that FAST is a nonstandard first-order theory. Currently only axioms for the *elimination* of the quantifiers from FAM have been given: this suffices for the deduction of theorems from FAST, necessary for the construction of sets. Further research may be directed further expanding the framework by formalizing axioms for the *introduction* of universal quantification over a family of variables indexed in a set X: such would allow the formulation of new theorems from FAST.

The cost-benefit analysis is this: the benefit is that FAST is a finite theory, which preserves the ontology of ZF, and which, besides EXT and the existential axioms EMPTY and INF, only has constructive axioms; the cost is that FAST because of its family set axiom entails a departure from the language of ZF. But as the latter merely means that set theory is then no longer a strict application of standard first-order logic, we say that the benefit exceeds the cost. That is, we conclude that FAST, because of its finiteness, is more elegant than ZF.

A Appendix: on second-order semantics

The objection to FAST has been raised, that it uses second-order semantics. In this appendix we argue, however, that it doesn't.

Consider the axiom PAIR of ZF, which we formulate using the abbreviation (2):

$$\forall x_1, x_2 \exists y \forall z(z \in y \Leftrightarrow z = x_1 \vee z = x_2) \tag{6}$$

No one would claim that this requires second-order semantics. There are also infinitely many theorems of ZF, one for every integer $n > 2$, that can be proven using PAIR:

$$\forall x_1, \ldots, x_n \exists y \forall z(z \in y \Leftrightarrow z = x_1 \vee \ldots \vee z = x_n) \tag{7}$$

No one would claim that any of these requires second-order semantics either.

Now we use the definition of the syntax as given in Sect. 2. With the first ten axioms (i)/(x) we can construct numbers 0 and 1, e.g. by defining $0 := \emptyset$ and $1 := \mathcal{P}(\emptyset) = \{\emptyset\}$, as well as the set $\{0,1\}$, since $\{0,1\} = \mathcal{P}(1)$. The syntax thus says that we now have variables x_0 and x_1 ranging over sets. We now consider the following expression:

$$\forall (u_j)_{j\in\{0,1\}} \exists Z \forall y (y \in Z \Leftrightarrow \exists j \in \{0,1\}(y = u_j)) \tag{8}$$

This is a theorem of FAM by EL1, but we view it in itself: consider the axiom system consisting of the axioms (i)/(x) mentioned in Sect. 2 plus formula (8). The latter is absolutely equivalent to PAIR, formula (6), so no one will claim that expression (8) or the axiom system requires second-order semantics. Likewise, no one would claim that any of the expressions

$$\forall (x_j)_{j\in\{1,\ldots,n\}} \exists Z \forall y (y \in Z \Leftrightarrow \exists j \in \{0,\ldots n\}(y = x_j)) \tag{9}$$

requires second-order semantics, as they are equivalent to theorems (7) of ZF. We now merely extend this to a countable infinity of variables:

$$\forall (x_j)_{j\in N} \exists Z \forall y (y \in Z \Leftrightarrow \exists j \in N(y = x_j)) \tag{10}$$

with $N := \{0,1,2,\ldots\}$ which is assumed to exist—cf. INF, axiom (v). In ZF this would already require a formula of infinite length, which is not permitted. But does formula (10) require second-order semantics? No, is doesn't: it requires as much second-order semantics as PAIR, which is none at all. And now we generalize this idea even further:

$$\forall X \forall (x_j)_{j\in X} \exists Z \forall y (y \in Z \Leftrightarrow \exists j \in X(y = x_j)) \tag{11}$$

This is FAM. Does this, then, require second–order semantics? No: formula (8) is equivalent to PAIR and thus doesn't use second-order semantics, and in none of the steps from (8) to (9) to (10) to (11) have we changed anything in the semantics, or introduced second-order semantics, or anything like that. Ergo, FAM doesn't use second-order semantics.

FAST is a matter of pure formalism. Starting with FAM, we get theorems by applying EL1: for each assignment of a value $\mathbf{X}$ to the variable X, we get theorems

$$\forall (u_j)_{j\in\mathbf{X}} \exists Z \forall y (y \in Z \Leftrightarrow \exists j \in \mathbf{X}(y = u_j)) \tag{12}$$

From there we get theorems by applying EL2: for each assignment of a value $\mathbf{u_x}$ to any of the variables $u_\mathbf{x}$ with $\mathbf{x} \in \mathbf{X}$ we get theorems

$$\exists Z \forall y (y \in Z \Leftrightarrow \exists j \in \mathbf{X}(y = \mathbf{u}_j)) \tag{13}$$

These are existential axioms of sets Z. This only uses first-order semantics: we have used nothing but FAM, the logical elimination axioms EL1 and EL2, and the assignment of a value to a variable as in first-order logic. If we talk about semantics, then let's compare it to PAIR of ZF. PAIR merely means that for every assignment of constant values to variables x_1 and x_2, there is a set that contains these constants. FAM merely means that for every assignment of a constant value $\mathbf{u_x}$ to any variable $u_\mathbf{x}$ with a label $\mathbf{x} \in \mathbf{X}$, there is, for any such set $\mathbf{X}$, a set that contains these constants. No second-order semantic here.

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References

[1] D. Van Dalen, H.C. Doets, H.C.M. de Swart, *Sets: Naive, Axiomatic and Applied* (Pergamon Press, Oxford, 1978)

[2] R. Montague, *Contributions to the Axiomatic Foundations of Set Theory*, dissertation, University of California, Berkeley (1957)

[3] E. Mendelson, *An Introduction to Mathematical Logic*, 4th ed. (Chapman & Hall, London, 1997)

[4] F.A. Muller, Log. Anal. **213**, 31-48 (2011)

[5] P. Bernays, *Axiomatic Set Theory* (Dover Publications Inc., Mineola, 1968)