Special Relativity in a pseudo-Riemannian 5-manifold with a curled-up fifth dimension

Marcoen J.T.F. Cabbolet*

Center for Logic and Philosophy of Science

Vrije Universiteit Brussel

Abstract — This paper is a self-contained introduction of Special Relativity (SR) in a pseudo-Riemannian 5-manifold with one curled-up dimension. Although presented here as a theory on its own, this five-dimensional account of SR is an integral part of the proof that the Elementary Process Theory corresponds to SR, see 'Relativity of Spatiotemporal Characteristics of Inertial Motion in the Universe of the Elementary Process Theory' by M. Cabbolet.

MSC: 83A05, 83E99
keywords: Special Relativity, 5-manifold, curled-up dimension

1 Introduction

Special Relativity (SR), first published in [1], is known for over a century now, and is widely considered to be an untouchable cornerstone of modern physics. This letter proves for a fact that SR is also possible in a five-dimensional (5D) space—more precisely, in a pseudo-Riemannian 5-manifold—with one curled-up dimension. The idea of a curled-up dimension is not new: it has been proposed earlier by Klein [2]. Nor is the idea of a 5D account of SR new: earlier 5D accounts of relativity (not limited to SR) can be found both in the pre-World-War-II literature, e.g. [3] and in the modern literature, e.g. [4, 5]. What is new here is that 5D SR implies that all particles—with or without rest mass—move with light speed in 4D space.

The outline of this paper is as follows. The next section introduces a pseudo-Riemannian 5-manifold. The section thereafter introduces SR in 5D spacetime; the final section proves that it is empirically equivalent to the 'standard' SR in 4D Minkowski spacetime, and states the conclusions. The remainder of this introduction is for some basic mathematical notations and definitions.

Notation 1.1 We will use the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ respectively for the sets of natural numbers, integers, rational numbers, real numbers, and complex numbers, and $\mathbb{U}$ for the unit interval $[0, 1) \subset \mathbb{R}$. The floor function $\lfloor . \rfloor : \mathbb{R} \to \mathbb{Z}$ is given by $\lfloor x \rfloor = \max\{y \in \mathbb{Z} \mid y \leq x\}$. For any $x \in \mathbb{R}$ and $y \in \mathbb{U}$, the expression $x \equiv y \pmod 1$ means: x is congruent to y modulo 1, that is, $|x - y| \in \mathbb{N}$. □

Definition 1.2 The space $(\mathbb{U}, +_1, \cdot)$ consists of the set $\mathbb{U}$ together with the binary operations addition $+_1 : \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ and scalar multiplication $\cdot : \mathbb{R} \times \mathbb{U} \to \mathbb{U}$, which are defined as follows for any $a, b, c \in \mathbb{U}$ and $x \in \mathbb{R}$:

$$a +_1 b = c \Leftrightarrow (a + b) \equiv c \pmod 1 \tag{1}$$

$$x \cdot a = b \Leftrightarrow xa \equiv b \pmod 1 \tag{2}$$

where the term 'xa' in Eq. (2) refers to the product $xa \in \mathbb{R}$. □

*e-mail: Marcoen.Cabbolet@vub.ac.be

The binary structure $(\mathbb{U}, +_1)$ is isomorphic to the abelian group $(S^1, \cdot)$, the set $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ of complex numbers of norm 1 under multiplication. $(\mathbb{U}, +_1)$ is thus an abelian group; for any $a \in \mathbb{U}$, we have thus an element '$-a$' such that $a +_1 (-a) = 0$. Furthermore, $1 \cdot a = a$ for any $a \in \mathbb{U}$.

Corollary 1.3 The space $(\mathbb{U}, +_1, \cdot)$ is not a vector space.

Proof A vector space must satisfy the axiom $\alpha \cdot (a+b) = \alpha \cdot a + \alpha \cdot b$ for any scalar α and any vectors a and b. However, here we have $\frac{3}{2} \cdot (\frac{1}{2} +_1 \frac{1}{2}) = \frac{3}{2} \cdot 0 = 0$ but $\frac{3}{2} \cdot \frac{1}{2} +_1 \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} +_1 \frac{3}{4} = \frac{1}{2}$. Q.E.D. □

Definition 1.4 Let $(\mathbb{R}, T)$ be the real line in the standard topology. Let the function $\varrho : \mathbb{R} \to \mathbb{U}$ be given by $\varrho : x \mapsto y \Leftrightarrow x \equiv y \pmod 1$. Then the **standard topology** on $\mathbb{U}$, notation: $T_{\mathbb{U}}$, is the topology coinduced on $\mathbb{U}$ by ϱ. Note that ϱ does not yield a *compactification*, because ϱ is not injective. We will use the function ϱ throughout this paper. □

Corollary 1.5 For any $x \in \mathbb{U}$, the set $\mathbb{U} \backslash \{x\}$ is open in $\mathbb{U}$.

Proof Consider the open subset $(x-1, x)$ of $\mathbb{R}$. Then by Def. 1.4, $\varrho[(x-1, x)]$ is open in $\mathbb{U}$. But $\varrho[(x-1,x)] = [0, x) \cup (x, 1) = U \backslash \{x\}$ as requested. Q.e.d. □

The space $(\mathbb{U}, T_{\mathbb{U}})$ is homeomorphic to the circle S^1 in its standard topology: $(\mathbb{U}, T_{\mathbb{U}})$ is thus a path connected, compact Hausdorff space.

2 Definition of a pseudo-Riemannian 5-manifold

Definition 2.1 The **pseudo-Riemannian 5-manifold** $(\mathcal{M}, T_{\mathcal{M}}, A_{\mathcal{M}}, T\mathcal{M}, g)$ consists of the set $\mathcal{M} = \mathbb{R}^4 \times \mathbb{U}$, the standard product topology $T_{\mathcal{M}}$, the atlas $A_{\mathcal{M}}$, the tangent bundle $T\mathcal{M}$, and the metric tensor field g, such that

(i) the atlas $A_{\mathcal{M}} = \{(U_t, \phi_t) \mid t \in \mathbb{Q}\}$ contains for every $t \in \mathbb{Q}$ a **chart** (U_t, ϕ_t) given by

$$\begin{array}{l} U_t = (t, t+1) \times \mathbb{R}^3 \times \mathbb{U} \backslash \{\varrho(t)\} \\ \phi_t : U_t \to \mathbb{R}^5 \ , \ \phi_t : \left\{ \begin{array}{lll} (x^0, x^1, x^2, x^3, x^4) \mapsto (x^0, x^1, x^2, x^3, \lfloor t+1 \rfloor + x^4) & if & x^4 \in [0, \varrho(t)) \\ (x^0, x^1, x^2, x^3, x^4) \mapsto (x^0, x^1, x^2, x^3, \lfloor t \rfloor + x^4) & if & x^4 \in (\varrho(t), 1) \end{array} \right. \end{array} \tag{3}$$

where ϱ is the function from Def. 1.4, and $[0, \varrho(t)) = \emptyset$ for $t \in \mathbb{Z}$. See Fig. 1 for an illustration.

(ii) the tangent bundle $T\mathcal{M}$ is the sum set $\bigcup\{T_P(\mathcal{M}) \mid P \in \mathcal{M}\} = \mathcal{M} \times \mathbb{R}^5$, where $T_P(\mathcal{M})$ is the **tangent space** at the point $P \in \mathcal{M}$ given by

$$T_P(\mathcal{M}) = \{P\} \times \mathbb{R}^5 = \{(p^0, p^1, p^2, p^3, p^4, x^0, x^1, x^2, x^3, x^4) \mid x^j \in \mathbb{R}\} \subset \mathbb{R}^{10} \tag{4}$$

If we denote $(p^0, p^1, p^2, p^3, p^4, x^0, x^1, x^2, x^3, x^4) \in T_P(\mathcal{M})$ as $\vec{x}_P$, with $\vec{x}_P = (x^0, x^1, x^2, x^3, x^4)_P$, and if we endow $T_P(\mathcal{M})$ with the operations addition and scalar multiplication given by

$$\vec{x}_P + \vec{y}_P = (x^0 + y^0, x^1 + y^1, x^2 + y^2, x^3 + y^3, x^4 + y^4)_P \tag{5}$$

$$\alpha \cdot \vec{x}_P = (\alpha x^0, \alpha x^1, \alpha x^2, \alpha x^3, \alpha x^4)_P \tag{6}$$

then $(T_P(\mathcal{M}), +, \cdot)$ is isomorphic to the standard five-dimensional vector space $(\mathbb{R}^5, +, \cdot)$.

(iii) the **tensor field** g adds to every point $P \in \mathcal{M}$ a metric tensor $g_P : T_P(\mathcal{M}) \times T_P(\mathcal{M}) \to \mathbb{R}$ such that for any two vectors $\vec{x}_P, \vec{y}_P \in T_P(\mathcal{M})$

$$g_P(\vec{x}_P, \vec{y}_P) = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4 = \eta^{(5)}_{ij} x^i y^j \tag{7}$$

where x^i, y^j are the coordinates with respect to a standard basis of $T_P(\mathcal{M})$. □

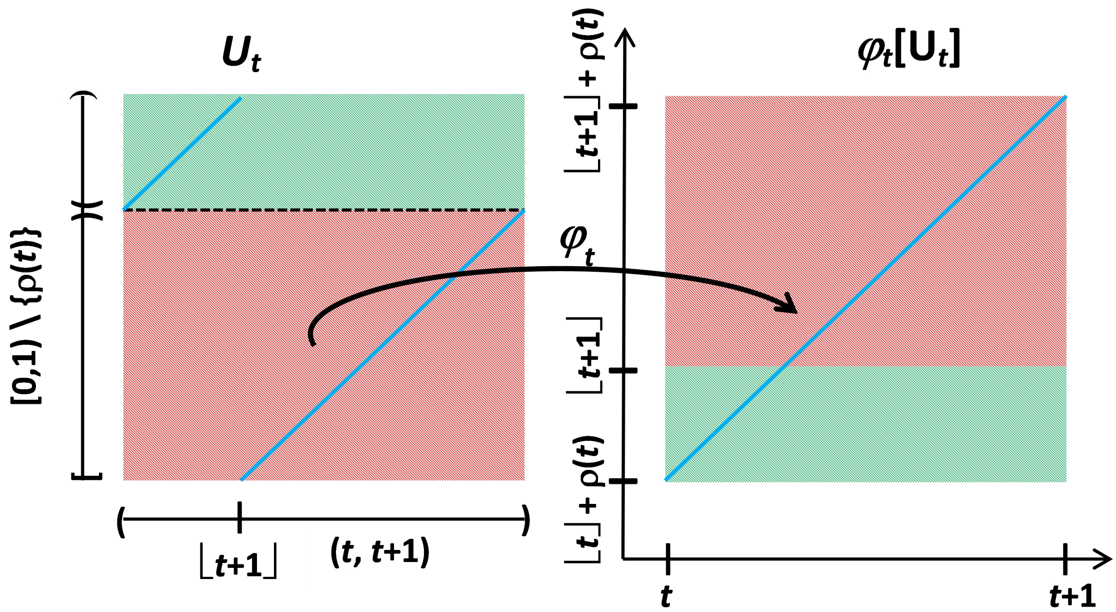

Figure 1: Illustration of the chart (U_t, ϕ_t) in the atlas $A_{\mathcal{M}}$. Suppressing $\mathbb{R}^3$, the square to the left represents $U_t = (t, t+1) \times \mathbb{R}^3 \times \mathbb{U}\backslash\{\varrho(t)\}$; the red part is $(t, t+1) \times [0, \varrho(t))$, the green part is $(t, t+1) \times (\varrho(t), 1)$. The blue line is the line ℓ in $\mathcal{M}$ with parametrization $(u, 0, 0, 0, \varrho(u))$. The square to the right represents $\phi_t[U_t]$, the image of U_t under ϕ_t. The red part, the green part and the blue line in the right square are the images of the corresponding items in the left square. The vertical axis of the right figure shows the total distance traveled in the curled-up dimension since $t = 0$ by an observer moving on ℓ (see Sect. 3); if $t = 0$ coincides with the beginning of the universe, then an integer value is the degree of evolution of the EPT that the observer is at (see Rem.4.7).

Definition 2.2 The binary operation **scalar multiplication** $\cdot : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ is for any number $a \in \mathbb{R}$ and any $X = (x^0, x^1, x^2, x^3, x^4) \in \mathcal{M}$ given by

$$a \cdot (x^0, x^1, x^2, x^3, x^4) = (ax^0, ax^1, ax^2, ax^3, a \cdot x^4) \tag{8}$$

where the product $a \cdot x^4$ is given by Def. 1.2. □

Definition 2.3 The $\mathcal{O}$**-group** $\mathcal{M}$ $(\mathcal{M}, +, \mathcal{O})$ consists of the set $\mathcal{M}$, given in Def. 2.1, the binary operations addition $+ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and the set of operators $\mathcal{O}$, such that

(i) the **sum** of any two elements $X = (x^0, x^1, x^2, x^3, x^4)$ and $Y = (y^0, y^1, y^2, y^3, y^4, y^5)$ of $\mathcal{M}$ is given by

$$(x^0, x^1, x^2, x^3, x^4) + (y^0, y^1, y^2, y^3, y^4) = (x^0 + y^0, x^1 + y^1, x^2 + y^2, x^3 + y^3, x^4 +_1 y^4) \tag{9}$$

where the sum $x^4 +_1 y^4$ is given by Def. 1.2. Note that $(\mathcal{M}, +)$ is thus an abelian group.

(ii) for every Lorentz transformation Λ with matrix $\begin{pmatrix} \lambda_{00} & \lambda_{01} & \lambda_{02} & \lambda_{03} \\ \lambda_{10} & \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{20} & \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{30} & \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$ on Minkowski space with signature $(-, +, +, +)$ there is an operator $\lambda \in \mathcal{O}$ with matrix $\begin{pmatrix} \lambda_{00} & \lambda_{01} & \lambda_{02} & \lambda_{03} & 0 \\ \lambda_{10} & \lambda_{11} & \lambda_{12} & \lambda_{13} & 0 \\ \lambda_{20} & \lambda_{21} & \lambda_{22} & \lambda_{23} & 0 \\ \lambda_{30} & \lambda_{31} & \lambda_{32} & \lambda_{33} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

so that

$$\lambda(X) = Y \Leftrightarrow \begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = \begin{pmatrix} \lambda_{00} & \lambda_{01} & \lambda_{02} & \lambda_{03} & 0 \\ \lambda_{10} & \lambda_{11} & \lambda_{12} & \lambda_{13} & 0 \\ \lambda_{20} & \lambda_{21} & \lambda_{22} & \lambda_{23} & 0 \\ \lambda_{30} & \lambda_{31} & \lambda_{32} & \lambda_{33} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \tag{10}$$

□

Remark 2.4 The space $(\mathcal{M}, +, \cdot)$ is **not** a vector space, for the same reason as mentioned in the proof of Cor. 1.3: it is **not** for any $\alpha \in \mathbb{R}$ and $X, Y \in \mathcal{M}$ the case that $\alpha \cdot (X + Y) = \alpha \cdot X + \alpha \cdot Y$. However, for any operator $\lambda \in \mathcal{O}$ and $X, Y \in \mathcal{M}$ we **do** get that $\lambda(X + Y) = \lambda(X) + \lambda(Y)$. □

Definition 2.5 A **null line** is a curve $\mathscr{C} \subset \mathcal{M}$ parameterized by $u \in \mathbb{R}$, such that the tangent $\vec{v}(u)_{X(u)} \in T_{X(u)}(\mathcal{M})$ of the path $\mathscr{C}$ at the point $X(u) \in \mathcal{M}$ satisfies

$$g_{X(u)}(\vec{v}(u)_{X(u)}, \vec{v}(u)_{X(u)}) = 0 \tag{11}$$

at any point $X(u)$ on the curve $\mathscr{C}$. □

Corollary 2.6 Let $\mathscr{C}$ be any null line in $\mathcal{M}$, and let $\lambda \in \mathcal{O}$. Then $\lambda[\mathscr{C}]$ is a null line.

Proof The proof is omitted. □

3 5D SR

Agreement 3.1 (Units) In the remainder of this text we will use **Planck units**: both Planck length and Planck time are scaled to 1. Furthermore, for the sake of simplicity we will use rectangular coordinates so that we can use the metric tensor $\eta^{(5)}$ of Eq. (7). □

Definition 3.2 (5D IRF) The **reference frame of an inertial observer in five-dimensional spacetime** is the manifold $\mathcal{M}$ of Def. 2.1. Such an inertial reference frame will henceforth be referred to by the acronym '5D IRF'. For a point $X = (x^0, x^1, x^2, x^3, x^4) \in \mathcal{M}$, the real number x^0 is the time coordinate, the three real numbers x^1, x^2, x^3 are the "regular" spatial coordinates, and the real number x^4 represents the coordinate in the curled-up fifth dimension. We will call this number $x^4 \in \mathbb{U}$ simply the *fourth spatial coordinate.* □

A **presupposition** of this definition is that curvature can be neglected. We shall take this to mean that this 5D SR only applies when interactions can be neglected. To put that more precisely: let the **five-velocity** of a particle at a point $X(t)$ be the tangent vector $\vec{v}_{X(t)} = (1, v^1, v^2, v^3, v^4)_{X(t)} \in T_{X(t)}(\mathcal{M})$ of its 5D world line $\mathscr{C}$ parameterized by its time coordinate t; then we presuppose that for any particle, the numbers v^j of the five-velocity $(1, v^1, v^2, v^3, v^4)_{X(t)}$ are constant, that is, are independent of t. In other words: this 5D SR only applies when the motion of particles can be modeled by linear motion.

Notation 3.3 Let $\mathcal{E}$ be a singular event, i.e., an event that corresponds to a single point in a 5D IRF. Taken from [6], the expression

$$\mathcal{E} \underset{\mathcal{O}}{\longrightarrow} (t, x, y, z, n) \tag{12}$$

has then to be read as: in the 5D IRF of observer $\mathcal{O}$, the event $\mathcal{E}$ has coordinates (t, x, y, z, n). Furthermore, if X is a point in a 5D IRF, then $(X)^j$ is its j^{th} coordinate. So if $X = (t, x, y, z, n)$, then $(X)^4 = n$ and $(X)^0 = t$. We will use the symbol $\vec{v}^{(5)}_{X(t)}$ for the five-velocity at the point $X(t)$. □

We now give three postulates, that together constitute 5D SR—but just as there are several equivalent ways to formulate 'standard' SR, there may be other equivalent formulations possible.

Postulate 3.4 There is no preferred 5D IRF. □

Postulate 3.5 For any inertial observer $\mathcal{O}$ the 5D world line of any particle is a null line in the 5D IRF of $\mathcal{O}$, so that we have

$$\eta^{(5)}(\vec{v}^{(5)}_{X(t)}, \vec{v}^{(5)}_{X(t)}) = 0 \tag{13}$$

at any point $X(t)$ of any particle's 5D world line. □

Postulate 3.6 For any particle, the total distance traveled in the curled-up fifth dimension between any two events $\mathcal{E}_1$ and $\mathcal{E}_2$ on its 5D world line is observer-invariant. That is, for any inertial observer $\mathcal{O}$, for any two events $\mathcal{E}_1 \underset{\mathcal{O}}{\longrightarrow} (t_1, x_1, y_1, z_1, n_1)$ and $\mathcal{E}_2 \underset{\mathcal{O}}{\longrightarrow} (t_2, x_2, y_2, z_2, n_2)$ on any particle's 5D world line ℓ with $t_2 > t_1$, the number

$$\Delta n = \int_{t_1}^{t_2} |v^4| dt \tag{14}$$

is an observer-independent constant—here v^4 is a coordinate of $\vec{v}^{(5)}_{X(t)} = (1, v^1, v^2, v^3, v^4)_{X(t)}$. □

We proceed by stating a surprising theorem of 5D SR:

Theorem 3.7 (Universality of particle speed in 5D SR)
For any inertial observer $\mathcal{O}$, all particles move with the speed of light.

Proof This follows straight from Post. 3.5. For any particle we have $\vec{v}^{(5)}_{X(t)} = (1, v^1, v^2, v^3, v^4)_{X(t)}$, so Eq. (13) yields

$$(v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2 = 1 \tag{15}$$

meaning that any particle moves with light speed in **four-dimensional** space. Q.E.D. □

4 Correspondence to 'standard' SR

We now derive the two postulates of 'standard' SR from 5D SR. To formulate these neatly, we define the **3-speed** of a particle with five-velocity $(1, v^1, v^2, v^3, v^4)_{X(t)}$ at a point $X(t)$ in the 5D IRF of an inertial observer $\mathcal{O}$ to be the number

$$v_{(3)} = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2} \tag{16}$$

Theorem 4.1 (Principle of universality of the 3-speed of light)
The 3-speed of light, c, is a universal constant that has the same vale 1 for any inertial observer $\mathcal{O}$.

Proof Consider an observer $\mathcal{O}$, who has observed the 3-speed of a photon to be 1 between two events $\mathcal{E}_1$ and $\mathcal{E}_2$. Given Eq. (15) of Th. 3.7, we thus have that the component v^4 of the five-velocity of the photon is 0 in the 5D IRF of $\mathcal{O}$. But that means that the distance traveled by the photon in the curled-up fifth dimension is zero: the invariant number $\Delta n = 0$, cf. Eq. (14). But by Post. 3.6, that means that $\Delta n = 0$ for the photon in the 5D IRF of *any* observer. Ergo, the component v^4 of the five-velocity of the photon is 0 in the 5D IRF of any observer; we thus have that $v_{(3)} = 1$ for any photon for any observer. □

Theorem 4.2 (Principle of relativity in 5D SR)
No experiment can determine the absolute 3-speed of an observer.

Proof What we have to prove is that time passes at a different rate for observers that are not co-moving. Consider two events $\mathcal{E}_1$ and $\mathcal{E}_2$ on the 5D world line of a rest-mass-having particle in the 5D IRF of a co-moving inertial observer $\mathcal{O}$. We thus have $v_{(3)} = 0$ for the particle in the 5D IRF of $\mathcal{O}$, which leaves for its five-velocity $\vec{v}^{(5)}_{X(t)} = (1, v^1, v^2, v^3, v^4)_{X(t)} = (1, 0, 0, 0, 1)_{X(t)}$ at any point $X(t)$ of the particle's 5D world line. Now let $\mathcal{E}_1 \underset{\mathcal{O}}{\longrightarrow} (t_1, x, y, z, n_1)$ and $\mathcal{E}_2 \underset{\mathcal{O}}{\longrightarrow} (t_2, x, y, z, n_2)$; we then get for the integral (14)

$$\Delta n = v^4(t_2 - t_1) = \Delta t \tag{17}$$

As this number is observer-invariant, cf. Post. 3.6, it must also be obtained in the 5D IRF of an inertial observer $\mathcal{O}'$ where the 3-speed of that particle satisfies $v'_{(3)} > 0$. But Eq. (13) of Post. 3.5 then yields that $v^{4\prime} < 1$ for the coordinate $v^{4\prime}$ of the particle's five-velocity. Now let $\mathcal{E}_1 \underset{\mathcal{O}'}{\longrightarrow} (t'_1, x'_1, y'_1, z'_1, n'_1)$ and $\mathcal{E}_2 \underset{\mathcal{O}'}{\longrightarrow} (t'_2, x'_2, y'_2, z'_2, n'_2)$; for the above number Δn we thus get

$$\Delta n = v^{4\prime}(t'_2 - t'_1) = v^{4\prime}\Delta t' \tag{18}$$

But since $v^{4\prime} < 1$, we get $\Delta t' > \Delta t$. That means that between the events $\mathcal{E}_1$ and $\mathcal{E}_2$ time has passed at a different rate for observers $\mathcal{O}$ and $\mathcal{O}'$. Q.E.D. $\square$

The 'standard' theory of SR can be derived from the above Ths. 4.1 and 4.2 [6]. The following lemmas of 5D SR we state without prove:

Lemma 4.3 (5D Lorentz transformation)
For any two inertial observers $\mathcal{O}$ and $\mathcal{O}'$, there is an operator λ in the set $\mathcal{O}$ of Def. 2.3, such that λ is the coordinate transformation from the 5D IRF of $\mathcal{O}$ to the 5D IRF of $\mathcal{O}'$ (provided the origins coincide). $\square$

Lemma 4.4 (Correspondence to 4D SR)
Under the projection $\pi : \mathcal{M} \to \mathbb{R}^4$, $\pi : (t, x, y, z, n) \mapsto (t, x, y, z)$ the null lines of particles in the 5D IRF of an observer $\mathcal{O}$ yield the world lines of the particles in the 'usual' 4D Minkowski space with a metric g with signature $(-, +, +, +)$. $\square$

Lemma 4.5 (Interpretation of the 'invariant interval' in 5D SR)
Let, π be the projection of Lemma 4.4, and let for any observer $\mathcal{O}$, $\mathcal{E}_1 \underset{\mathcal{O}}{\longrightarrow} (t_1, x_1, y_1, z_1, n_1)$ and $\mathcal{E}_2 \underset{\mathcal{O}}{\longrightarrow} (t_2, x_2, y_2, z_2, n_2)$ be any two events on the 5D world line of any particle in the 5D IRF of $\mathcal{O}$, with $t_2 > t_1$; then the invariant interval Δs of the corresponding 4D spatiotemporal displacement vector $\Delta\vec{x} = \pi((t_2, x_2, y_2, z_2, n_2)) - \pi((t_1, x_1, y_1, z_1, n_1))$, for which

$$\Delta s = \sqrt{-g(\Delta\vec{x}, \Delta\vec{x})} \tag{19}$$

is identical the total distance Δn that the particle has traveled in the curled-up fifth dimension in the 5D IRF of $\mathcal{O}$, cf. Post. 3.6. $\square$

Lemma 4.6 (Philosophy of Time)
For any observer $\mathcal{O}$ and for any two events $\mathcal{E}_1 \underset{\mathcal{O}}{\longrightarrow} (t_1, x_1, y_1, z_1, n_1)$ and $\mathcal{E}_2 \underset{\mathcal{O}}{\longrightarrow} (t_2, x_2, y_2, z_2, n_2)$ on the 5D world line of any particle in the 5D IRF of $\mathcal{O}$, with $t_2 > t_1$, the duration of the particles displacement $\Delta t = t_2 - t_1$ is nothing but the Euclidean measure of the displacement in 3D space and the total distance Δn that the particle has traveled in the curled-up fifth dimension in the 5D IRF of $\mathcal{O}$:

$$\Delta t = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 + \Delta n^2} \tag{20}$$

where $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, and $\Delta z = z_2 - z_1$. $\square$

Remark 4.7 (5D SR as the physical meaning of the EPT)
The Elementary Process Theory (EPT) consists of seven well-formed formulas (wffs) together with a physical interpretation [7, 8]. This yields the picture that these seven wffs describe for integers k and n what happens in the k^{th} process from the n^{th} to the $(n+1)^{\text{th}}$ degree of evolution, which is a *generic individual process* that takes place at supersmall scale in the universe of the EPT: we thus have the parameter 'degrees of evolution' in the EPT. In [9] it has been shown by an analysis of the generic process that the invariance of the squared interval of SR *emerges from* the EPT if we view the degrees of evolution as an additional dimension that can be modeled by the set $\mathbb{R}$ together with an equivalent relation $\sim$, given by

$$x \sim y \Leftrightarrow x \equiv y (\text{mod } 1) \tag{21}$$

where $x \sim y$ has to be interpreted as 'x and y are *physically* the same point'. As there is a natural bijection $\phi : [x]_\sim \leftrightarrow x$ between the cells $[x]_\sim$ of the partitioning of $\mathbb{R}$ induced by $\sim$ and the elements $x \in \mathbb{U}$, 5D SR can be seen as a **concrete mathematical model** of aspects of the abstractly formulated EPT, which has to be understood as follows:

(i) the curled-up fifth dimension of $\mathcal{M}$ is a **model of the set of physically distinct points** in the dimension of degrees of evolution;

(ii) the 5D world line of a rest-mass-having particle is a **continuous approximation model** of the successive definite positions attained by a rest-mass-having particle in the universe of the EPT;[2]

(iii) the 5D world line of zero-rest-mass particles in 5D SR is a **dimensionless particle model** of the motion of zero-rest-mass matter quanta in the universe of the EPT; for any position X on any such world line, we have $(X)^4 = 0$.

5D SR thus describes the nature of spacetime in the universe of the EPT under special-relativistic conditions. □

Concluding, a 5D SR has been rigorously introduced, and it has been shown that it corresponds to 'standard' SR: 5D SR is thus *empirically equivalent* to 4D SR. Although a theory on its own, 5D SR gives, as a model of the EPT, a concrete physical meaning to the abstract EPT.

References

[1] A. Einstein, *Ann. Phys. (Leipzig)* **17**, 891-921 (1905)

[2] O. Klein, *ZS f. Physik* **37**, 895-906 (1926)

[3] T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.)*: 966-972. (1921)

[4] C.J. Bordé, *Phil. Trans. Roy. Soc. A* **363**, 2177-2201 (2005)

[5] S. Capozziello, G. Basini, & M. De Laurentis, *Eur. Phys. J. C* **71**, 1679 (2011)

[6] B.F. Schutz, *A First Course in General Relativity*, Cambridge: Cambridge University Press (1990)

[7] M.J.T.F. Cabbolet, *Ann. Phys. (Berlin)* **522**, 699-738 (2010)

[8] M.J.T.F. Cabbolet, *Ann. Phys. (Berlin)* **523**, 990-994 (2011)

[9] M.J.T.F. Cabbolet, *How the Elementary Process Theory corresponds to Special Relativity: 'degrees of evolution' as a curled-up fifth dimension.* Preprint: philpapers.org/rec/CABHTE (2015)

[2] These exhibit stepwise motion in the EPT: each step is an observer-independent unit displacement in degrees of evolution [9]; in $\mathcal{M}$, this is a leap $X \to Y$ with $(X)^4 = (Y)^4 = 0$.